

THE KURATOWSKI CONVERGENCE OF MEDIAL AXES

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ABSTRACT. In this paper we study the behaviour of medial axes (skeletons) of closed, definable (in some o-minimal structure) sets in \mathbb{R}^n under deformations. We apply a new approach to the deformation process. Instead of seeing it as a ‘jump’ from the initial to the final state, we perceive it as a continuous process, expressed using the Kuratowski convergence of sets (hence, unlike other authors, we do not require any regularity of the deformation). Our main result has already proved useful, as it was used to compute the tangent cone of the medial axis with application in singularity theory.

1. INTRODUCTION

It has been known for a long time that the *medial axis* ⁽¹⁾ is highly unstable under small deformations. In particular, F. Chazal and R. Soufflet gave in [3] a simple illustration of this fact: the medial axis of a circle is its central point, but even the smallest ‘protuberance’ on the circle leads to the medial axis becoming a whole segment. Their paper [3] is entirely devoted to showing that under some hypotheses on X there is a kind of stability of the medial axis under \mathcal{C}^2 deformations. Their approach consists in looking at the initial and the final steps only — with nothing in between, so to say.

In the present paper we adopt another, natural, point of view: we see the deformation as a continuous process that we do not even require to be smooth. This lets us have some insight of what is happening to the medial axis.

We have chosen the Kuratowski (or *Painlevé-Kuratowski*) convergence of closed sets as a method of approach to this process of deformation. In this way we have obtained our main Theorem 4.1 which has already proved useful: it is used in [1] in order to compute the tangent cone of the medial axis, which permitted to give some relation between

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¹As explained later, what we mean by *medial axis* of a given closed, nonempty set $X \subset \mathbb{R}^n$ is the set of points in \mathbb{R}^n for which there is more than one closest point to X with respect to the Euclidean distance.

the medial axis and the type of singularity of the given definable set (see Example 4.4).

The paper is organized as follows. After recalling the basic definitions (Section 2), we give a large range of examples to illustrate the obstacles encountered by mathematicians, us included, who believed that the problem of convergence of the medial axes was an easy one (Section 3). The examples lead to two natural conjectures that turn out to be false (as shown in Examples 3.5 and 3.11, respectively). We do this to show why our approach via the Kuratowski convergence seems to be the most appropriated and the result obtained in Theorem 4.1 is somehow optimal (cf. Remark 4.3 and Example 3.11). Although natural, it has eluded to specialists who had judged it at first elementary and easy. We have also been through this experience and it is our way of sharing it with the reader. We end the paper with a discussion of what could be a natural proof of our main result in the non-definable setting and why it does not seem to work (Remark 4.6). This last remark is a good illustration of how a seemingly easy proof becomes impossible to be completed when it comes to details.

2. PRELIMINARIES

Throughout the paper we study the behaviour of the medial axes of a family of closed subsets of \mathbb{R}^n in the setting of *tame geometry*. We are interested in families definable in some *o-minimal structure* over the field of real numbers (as in [8], [1]; for definable sets see e.g. [4] and also [7]). The medial axis is closely related to the notion of *central set* (the set of centres of maximal balls contained in $\mathbb{R}^n \setminus X$) and cut locus, and appears sometimes under the name of skeleton or cut locus (although this need not denote precisely the same concept). It plays an important role in pattern recognition (see [3] for references), but has applications also in variational analysis (historically [5] seems to be the first paper hinting at that, see also [10]) or singularity theory which is particularly of interest to us (cf. [13], [2], [8], [1]).

2.1. Medial axis and central set. We recall the basic definitions. For a given closed, nonempty set $X \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we shall write

$$m(x) = \{y \in X \mid \|x - y\| = \text{dist}(x, X)\}$$

for the set of points in X realizing the Euclidean distance $\text{dist}(x, X)$ of x to X ('the closest points'). Then the *medial axis* of X is defined to be the set

$$M_X := \{x \in \mathbb{R}^n \mid \#m(x) > 1\}$$

i.e. the set of points where the multifunction $x \mapsto m(x)$ is not univalent.

Recall that an open ball $\mathbb{B}(x, r) \subset \mathbb{R}^n \setminus X$ is said to be *maximal* if it is not contained in any other ball in $\mathbb{R}^n \setminus X$ as a proper subset. Then the *central set* of X , denoted by C_X , is defined as the set of centres of

all maximal balls in $\mathbb{R}^n \setminus X$. As observed in [3] (for a proof see e.g. [1]), we have

$$M_X \subset C_X \subset \overline{M_X}.$$

2.2. Kuratowski convergence. For set convergence we refer the reader the excellent book [11]. Here we adopt the point of view of [6].

In what follows we will consider a set $X \subset \mathbb{R}^k \times \mathbb{R}^n$ in the variables (t, x) where t is considered to be a parameter. We shall write X_t for the t -sections $X_t = \{x \in \mathbb{R}^n \mid (t, x) \in X\}$. Consider the projection $\pi(t, x) = t$ and assume that $0 \in \overline{\pi(X) \setminus \{0\}}$. We define the *upper* and the *lower Kuratowski limits* of X_t for $\pi(X) \setminus \{0\} \ni t \rightarrow 0$ as in [6]:

- $x \in \limsup X_t$ iff for any neighbourhoods $U \ni x, V \ni 0$, there is a parameter $t \in \pi(X) \cap V \setminus \{0\}$ such that $X_t \cap U \neq \emptyset$;
- $x \in \liminf X_t$ iff for any neighbourhood $U \ni x$, there is a neighbourhood $V \ni 0$ such that for any $t \in \pi(X) \cap V \setminus \{0\}$, we have $X_t \cap U \neq \emptyset$.

As $\liminf X_t \subset \limsup X_t$, we have convergence iff the converse inclusion holds. In particular, X_t converges to X_0 when $t \rightarrow 0$, which we denote by $X_t \xrightarrow{K} X_0$, iff

$$\limsup X_t \subset X_0 \subset \liminf X_t.$$

Recall that the upper and lower limits are always closed sets and do not change if we compute them for the closures $\overline{X_t}$.

It is worth noting that $X_0 = \limsup X_t$ iff for any compact set K disjoint with X_0 , there is a neighbourhood V of $0 \in \mathbb{R}^n$ such that $X_t \cap K = \emptyset$ for all $t \in V$ (see e.g. [6]).

In what follows we will restrict ourselves to the situation when $X \subset \mathbb{R}^k \times \mathbb{R}^n$ is a definable set (i.e. a definable family X_t , or, in other words, a definable multifunction). Here *definable* means *definable in some o-minimal structure* — we refer the reader to [4], but also [7] in order to see how this is related to subanalytic sets.

2.3. Some notation. For $X \subset \mathbb{R}^k \times \mathbb{R}^n$ with closed t -sections we will use the notation

$$m(t, x) = \{(t, y) \in \mathbb{R}^k \times \mathbb{R}^n \mid (t, y) \in X : \|x - y\| = \text{dist}(x, X_t)\}$$

and $m_t(x) = m(t, x)$. Then the medial axes M_{X_t} correspond to the sections M_t of the set

$$M := \{(t, x) \in \mathbb{R}^k \times \mathbb{R}^n \mid \#m(t, x) > 1\}.$$

All these sets are definable in the definable case, see [8]. For a fixed t , m_t is univalent apart from the set M_t .

3. INTRODUCTORY EXAMPLES

Consider a closed (definable), nonempty set $X \subset \mathbb{R}^k \times \mathbb{R}^n$ in the variables (t, x) . We always assume that $t = 0$ is an accumulation point of $\pi(X)$.

If we put $\delta_t(x) = \text{dist}(x, X_t)^2$, then we know that

- (1) δ_t is (definable), locally Lipschitz, vanishing exactly on X_t ;
- (2) the non-differentiability points of δ_t coincide with M_t (see [13], [1]);
- (3) $(t, x) \mapsto \delta_t(x)$ is continuous provided the sections X_t vary continuously (see [9]).

Of course, there is no relation whatsoever between the medial axis of X and the medial axes of the sections. But from [8] we know that $m(t, x)$ is a definable multifunction i.e. its graph $\Gamma_m = \{(t, x, y) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \mid y \in m_t(x)\}$ is definable. Clearly the graph Γ_{m_t} of m_t is the t -section of Γ_m .

Since we assume here X to be closed, we have $\limsup X_t \subset X_0$ (see [6]). Note that M_0 may be empty while $\limsup M_t$ is not:

Example 3.1. Let $X = \{(t, x, y) \in \mathbb{R}^3 \mid x^2 + y^2 = t^2\}$. Then the circles $X_t \xrightarrow{K} X_0 = \{(0, 0)\}$, but $M_t = \{(0, 0)\}$ for $t \neq 0$, whereas $M_0 = \emptyset$.

Note that in the Example above the dimension of X_t is not preserved at the limit. Let us have a look at an Example of constant dimension:

Example 3.2. Consider $X = \{(t, x, y) \in \mathbb{R}^3 \mid t^2 y = x^2 - 1\}$. Then for $t \neq 0$ each X_t is a parabola $y_t(x) = (1/t^2)(x^2 - 1)$, whereas $X_0 = \{-1, 1\} \times \mathbb{R}$. Clearly, $X_t \xrightarrow{K} X_0$.

It is easy to see that for any parabola X_t , $M_t = \{0\} \times (f_t, +\infty)$ ⁽²⁾ where $(0, f_t) = (0, y_t(0)) + \frac{1}{\kappa(0, y_t(0))} \nu(0, y_t(0))$ is the focal point of X_t , i.e. κ denotes the curvature and ν the unit normal which at $(0, y_t(0))$ is just $(0, 1)$. We compute the curvature as

$$\kappa_t(0) = \frac{y_t''(0)}{(1 + y_t'(0)^2)^{3/2}} = \frac{2}{t^2}$$

and so $f_t = \frac{t^2}{2} - \frac{1}{t^2} \rightarrow -\infty$ as $t \rightarrow 0$, i.e. ⁽³⁾ $M_t \xrightarrow{K} \{0\} \times \mathbb{R} = M_0$.

Unfortunately, the constancy of the dimension does not always guarantee the continuity of the medial axes:

Example 3.3. Let $X = \{(t, x, y) \in \mathbb{R}^3 \mid t^2 y = x^2\} \setminus \{(0, 0, y) \mid y < 0\}$. It is a closed semi-algebraic set with continuously varying t -sections:

²Note *en passant* that these sets are not closed in this example.

³We use the following known fact for plane curves, see [12]: if $\Gamma \subset \mathbb{R}^2$ is a \mathcal{C}^2 -smooth plane curve, and x lies on the normal to Γ at a , then a is the unique point realizing $d(x, \Gamma)$ iff the segment $[a, x]$ does not contain focal points; in particular, if $\|x - a\| < 1/\kappa(a)$ where $\kappa(a)$ is the curvature of Γ at a .

X_t is the parabola $y_t(x) = (1/t^2)x^2$ for $t \neq 0$ and the semi-line $\{0\} \times [0, +\infty)$ for $t = 0$. For each parabola we have $\kappa(0) = 2/t^2$ and so

$$M_t = \{0\} \times (t^2/2, +\infty) \xrightarrow{K} \{0\} \times [0, +\infty) \supsetneq M_0 = \emptyset.$$

Remark 3.4. In the last example X_0 has a singularity, whereas the nearby fibres X_t are smooth. It is important to observe that even in the definable setting, the smoothness of the limit does not necessarily imply the smoothness of the nearby sections ⁽⁴⁾:

$$X_t = \{(x, y) \in \mathbb{R}^2 \mid y = t|x|\} \xrightarrow{K} \{(x, 0) \mid x \in \mathbb{R}\} = X_0, \quad (t \rightarrow 0).$$

Neither does the smoothness of the sections X_t , together with connectedness and constancy of dimension guarantee the smoothness of the limit:

$$\begin{aligned} X_t &= \{(x, t) \in \mathbb{R}^2 \mid xy = t^2, x, y \geq 0\} \xrightarrow{K} \\ &\xrightarrow{K} \{(x, y) \in \mathbb{R}^2 \mid xy = 0, x, y \geq 0\} = X_0, \quad (t \rightarrow 0). \end{aligned}$$

These two examples are particularly interesting, since we have in both cases the best possible (from a set-theoretic point of view) situation: the sets in question are graphs converging locally uniformly ⁽⁵⁾.

Indeed, in the second case, after the obvious change of variables $u = (x - y)/2$, $v = (x + y)/2$, we have X_t described by $v^2 - u^2 = t^2$ in $\{(u, v) \mid v \geq |u|\}$. Thus, X_t is the (analytic) graph of the function $v(u) = \sqrt{u^2 + t^2}$. From its symmetry we infer, computing as earlier the curvature at the origin, that $M_t = \{0\} \times (2|t|, +\infty)$. So that $M_t \xrightarrow{K} \{0\} \times [0, +\infty) = \overline{M_0}$.

These examples lead to the following first natural conjecture. Note that in the definable setting it is often more natural to work with the upper limits (cf. [6]) which is why we start from this point of view.

Conjecture 1. *Assume that there is $X_0 = \limsup X_t$ for the definable set X , then $\limsup M_t \supset M_0$.*

However, it turns out immediately that Conjecture 1 is false:

Example 3.5. Consider the closed, semi-algebraic set $X \subset \mathbb{R} \times \mathbb{R}^2$ defined by the sections $X_0 := \{(x, y) \in \mathbb{R}^2 \mid y = |x|\}$ and $X_t = \{(x, y) \in \mathbb{R}^2 \mid y = \operatorname{sgn}(t)x, \operatorname{sgn}(t)x \geq 0\}$ for $t \neq 0$. Then $X_0 = \limsup X_t$, but $\liminf X_t = \{(0, 0)\}$ so that there is no convergence. We have $M_0 = \{0\} \times (0, +\infty)$, whereas for $t \neq 0$, $M_t = \emptyset$.

⁴In this case we also have M_t a constant family for $t \neq 0$, but $M_0 = \emptyset$.

⁵In particular, the convergence is without multiplicities, i.e. to each branch corresponds one branch in the converging sets. Note also that $v_\nu(u) = \sqrt{u^2 + (1/\nu)}$ is an example of a sequence of 1-Lipschitz \mathcal{C}^1 functions converging locally uniformly to a non-differentiable function.

We modify the conjecture:

Conjecture 2. *If X_t have a limit at 0 (not necessarily coinciding with X_0 , but at least included in it), then there exists also $\lim M_t$ and it contains M_0 .*

Remark 3.6. Note (cf. Example 3.8 hereafter) that the convergence of M_t is to be considered over $\pi(M)$.

If X_t does not converge, then in general neither M_t does:

Example 3.7. Let $X \subset \mathbb{R} \times \mathbb{R}^2$ be the closed semi-algebraic set defined by $X_0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$ and for $t \neq 0$, $X_t = \{(x, y) \in \mathbb{R}^2 \mid y = \operatorname{sgn}(t)|x|\}$. Then $X_0 = \limsup X_t$, $\liminf X_t = \{(0, 0)\}$ so that there is no convergence. Now, $M_t = \{0\} \times (0, +\infty)$ or $\{0\} \times (-\infty, 0)$ according to the sign of $t \neq 0$. Therefore,

$$\begin{aligned} \liminf M_t &= \{(0, 0)\} \subsetneq \limsup M_t = \\ &= \{0\} \times \mathbb{R} \subsetneq \overline{M_0} = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}. \end{aligned}$$

Observe that by the results of [6] we may assume that X has continuously varying sections (we lose, however, the closedness of X , since this assumption requires getting rid of a nowhere-dense subset of $\pi(X)$).

Example 3.8. Note that the assumptions:

$$X \text{ closed and definable, } X_0 = \lim X_t,$$

do not imply necessarily that the nearby sections are continuous. To see this consider two examples.

The first one is the subanalytic set $X = (\mathbb{R} \times \{0\}) \cup \bigcup_{n=1}^{+\infty} \{(1/n, n)\}$. Of course this is not definable.

The second one is a general semi-algebraic example but with two-dimensional parameters: $X = (\mathbb{R}^2 \times \{0\}) \cup \{(x, 0, x) \mid x \in \mathbb{R}\}$. By [6], apart from a nowhere-dense set in the parameters, all the sections are continuous. Note that in this example we have to throw away exactly those parameters over which the M_t 's are non-void.

Example 3.9. Consider X given by

$$X_t = \begin{cases} \mathbb{S}^1 \cup \{(0, 0)\}, & t > 0; \\ \mathbb{S}^1 \cup (1/2)\mathbb{S}^1, & t < 0; \\ \mathbb{S}^1 \cup (1/2)\mathbb{S}^1 \cup \{(0, 0)\}, & t = 0. \end{cases}$$

This is a closed semi-algebraic set with $X_0 = \limsup X_t$ and $\liminf X_t = \mathbb{S}^1$. Here

$$M_t = \begin{cases} (1/2)\mathbb{S}^1, & t > 0; \\ (3/4)\mathbb{S}^1 \cup \{(0, 0)\}, & t < 0; \\ (3/4)\mathbb{S}^1 \cup (1/4)\mathbb{S}^1, & t = 0 \end{cases}$$

and so $\liminf M_t = \emptyset$, $\limsup M_t = (1/2)\mathbb{S}^1 \cup (3/4)\mathbb{S}^1 \cup \{(0, 0)\}$ which shows that there is no relation whatsoever with M_0 .

One final Example to show that in Conjecture 2 the limit $\lim M_t$ depends on how the sets X_t converge rather than on the limit X_0 :

Example 3.10. Let $X_0 = \mathbb{R} \times \{0\}$, and consider

$$X_1 = ((-\infty, -1] \times \{0\}) \cup \{(x, -\operatorname{sgn}(x)x+1) \mid |x| \leq 1\} \cup ([1, +\infty) \times \{0\}).$$

The two simplest ways of making X_1 continuously evolve to X_0 are the following: we define for $t \in (0, 1)$, the sets X_t either as

$$((-\infty, -(1/t)] \times \{0\}) \cup \{(x, -t^2 \operatorname{sgn}(x)x+t) \mid |x| \leq 1\} \cup ((1/t), +\infty) \times \{0\})$$

or as

$$((-\infty, -1] \times \{0\}) \cup \{(x, -t \operatorname{sgn}(x)x + t) \mid |x| \leq 1\} \cup ([1, +\infty) \times \{0\}).$$

In both cases the sets M_t converge and the limit contains $\{0\} \times (-\infty, 0]$ (actually, it reduces to it in the first case), but in the second case it contains also $\{-1, 1\} \times [0, +\infty)$.

Now we give a definable counter-example to Conjecture 2:

Example 3.11. Consider the set $X = \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid y = t|x|\}$ from Remark 3.4. It is definable, we have $X_t \xrightarrow{K} X_0$, but

$$M_t = \begin{cases} \{(x, y) \mid x = 0, y > 0\}, & t > 0, \\ \emptyset, & t = 0, \\ \{(x, y) \mid x = 0, y < 0\}, & t < 0, \end{cases}$$

so that there is no convergence.

Observe that in all the previous examples with converging sections, we had $\liminf M_t \supset M_0$. This remark leads to the main theorem presented in the next section.

4. MAIN THEOREM

Theorem 4.1. *Assume that $X \subset \mathbb{R}_t^k \times \mathbb{R}_x^n$ is definable, has closed t -sections and $X_t \xrightarrow{K} X_0$. Then for $M = \{(t, x) \mid \#m(t, x) > 1\}$, we have*

$$\liminf_{\pi(M) \ni t \rightarrow 0} M_t \supset M_0$$

where we posit $\liminf M_t = \emptyset$ when $0 \notin \overline{\pi(M) \setminus \{0\}}$.

Remark 4.2. The Theorem implies that 0 cannot be an isolated point of $\pi(M) = \{t \mid M_t \neq \emptyset\}$, i.e. $M_0 = \emptyset$, if $0 \notin \overline{\pi(M) \setminus \{0\}}$.

Remark 4.3. Example 3.11 shows that we can hardly expect a better result even in the quite regular situation when we are dealing with a convergent definable one-parameter family of graphs.

Example 4.4. Before we prove this Theorem, let us give one important application used in [1]. In the definable setting, due to the Curve Selecting Lemma, the tangent cone $C_0(X)$ to a definable set $X \subset \mathbb{R}^n$ at $x = 0$ is obtained as the limit

$$C_0(X) = \lim_{t \rightarrow 0^+} (1/t)X.$$

In particular, if we know that $C_0(X)$ has a nonempty medial axis, then we conclude using the Theorem above, not only that $M_X \neq \emptyset$ but also that $0 \in \overline{M_X}$. Moreover, since it is obvious that $M_{(1/t)X} = (1/t)M_X$ and we are definable, we obtain even the convergence of the medial axes of the dilatations and the limit is clearly $C_0(M_X)$. The Theorem above gives then $C_0(M_X) \supset M_{C_0(X)}$.

We shall need the following simple Lemma.

Lemma 4.5. *Let $I \subset \mathbb{R} \times \mathbb{R}$ be a definable set in the variables (t, s) . Then the function $r: \mathbb{R} \ni t \mapsto \sup I_t \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is definable, too.*

Proof. The level set $\{t \mid r(t) = -\infty\}$ coincides with $\mathbb{R} \setminus \pi(I)$ where $\pi(t, s) = t$. We prove the definability of r over $\pi(I)$.

For $R < +\infty$,

$$r(t) = R \Leftrightarrow (\forall s \in I_t, s \leq R) \text{ and } (\forall \varepsilon > 0, \exists s \in I_t: R - \varepsilon < s)$$

and

$$r(t) = +\infty \Leftrightarrow \forall R > 0, \exists s \in I_t: s \geq R.$$

The latter shows that the level set $\{t \mid r(t) = +\infty\}$ is definable and the former proves that over its complement $r(t)$ is a definable function, which accounts for the definability of $r(t)$. \square

Proof of Theorem 4.1. Assume that there is a point

$$a \in M_0 \setminus \liminf M_t.$$

There are two cases to deal with depending on whether $t = 0$ is an accumulation point of $\pi(M)$, or not. We will treat them simultaneously.

If $0 \in \overline{\pi(M) \setminus \{0\}}$, this implies that for some ball B centred at a and for any neighbourhood $V \ni 0$, there is a point $t \in \pi(M) \cap V \setminus \{0\}$ such that $M_t \cap B = \emptyset$. In other words $t = 0$ belongs to the closure of the set

$$E := \{t \in \pi(M) \setminus \{0\} \mid M_t \cap B = \emptyset\}.$$

This set is clearly definable, hence by the Curve Selecting Lemma there is a \mathcal{C}^1 definable curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^k$ such that $\gamma(0) = 0$ and $\gamma((0, \varepsilon)) \subset E$. Then the set $X' := \{(\tau, x) \in [0, \varepsilon) \times \mathbb{R}^n \mid (\gamma(\tau), x) \in X\}$ is definable and satisfies $X'_\tau = X_{\gamma(\tau)} \xrightarrow{K} X'_0 = X_0$, when $\tau \rightarrow 0^+$.

On the other hand, if 0 is isolated in $\pi(M)$, then we may take X' to be for instance $X \cap ([0, \varepsilon) \times \{0\}^{k-1} \times \mathbb{R}^n)$ and we still have the

convergence of X'_τ to X_0 together with a neighbourhood of a in which there are no medial axes for non-zero parameters.

This means that we may restrict our considerations to a one-parameter definable family of converging sets $X_t \xrightarrow{K} X_0$ ($t \rightarrow 0^+$) and choose a neighbourhood $U \ni a$ such that all the m_t 's are univalent in U for $t \neq 0$. Put $d_t(x) := \text{dist}(x, X_t)$. Now, define

$$r(t) := \sup\{s \geq 1 \mid \mathbb{B}(m_t(a) + s(a - m_t(a)), s \cdot d_t(a)) \cap X_t = \emptyset\}, \quad t \neq 0,$$

i.e. we consider the open ball $\mathbb{B}(a, d_t(a))$ and start to inflate it from the point $m_t(a)$ in the direction $a - m_t(a)$ and keeping the tangency point $m_t(a)$ to X_t — we do this as long as possible without meeting X_t .

The function $r: (0, \varepsilon) \rightarrow [1, +\infty]$ is clearly definable. It follows from the description and the definability of the family X_t . Indeed, consider the complement of the set (for a fixed $t \in (0, \varepsilon)$)

$$\{s \geq 1 \mid \mathbb{B}(m_t(a) + s(a - m_t(a)), sd_t(a)) \cap X_t = \emptyset\}$$

— as it is described by the condition

$$\exists x \in \mathbb{B}(m_t(a) + s(a - m_t(a)), sd_t(a)) \cap X_t,$$

it can be written as the image

$$I_t := p(\{(x, s) \in \mathbb{R}^n \times [1, +\infty) \mid x \in \mathbb{B}(m_t(a) + s(a - m_t(a)), sd_t(a)) \cap X_t\})$$

under the projection $p(x, s) = s$. Now, let us introduce the sets $Y = X \times [1, +\infty)$ and

$$B = \{(t, x, s) \in (0, \varepsilon) \times \mathbb{R}^n \times [1, +\infty) \mid \|x - m_t(a) - s(a - m_t(a))\| < sd_t(a)\}.$$

Both are definable subsets of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$: in the case of Y it is obvious and for B it follows from the definability of the maps associating to t the number $d_t(a)$ and the vector $m_t(a)$, respectively. It remains to observe that

$$I_t = p((B \cap Y)_t)$$

where $(B \cap Y)_t = B_t \cap Y_t$ denotes the t -section. Finally, consider the projection $\varrho(t, x, s) = (t, s)$. Then

$$p((B \cap Y)_t) = (\varrho(B \cap Y))_t$$

which shows that the set

$$I := \{(t, s) \in (0, \varepsilon) \times [1, +\infty) \mid s \in I_t\} = \varrho(B \cap Y)$$

is definable and I_t is its t -section.

It remains to use Lemma 4.5 to conclude that $r(t)$ is definable. As such, it has a limit at 0^+ . Let us denote it by $r_0 := \lim_{t \rightarrow 0^+} r(t)$. A priori we could have $r_0 = +\infty$.

First, let us observe that the function associating to t the tangent planes to the spheres ⁽⁶⁾

$$\tau: (0, \varepsilon) \ni t \mapsto T_{m_t(a)} \partial \mathbb{B}(a, d_t(a)) \in G_{n-1}(\mathbb{R}^n)$$

is a definable curve and as such has a well-defined limit $T = \lim_{t \rightarrow 0+} \tau(t)$. Note that $\mu(t) := m_t(a)$ is again a definable curve, hence it admits a limit $y = \lim_{t \rightarrow 0+} \mu(t)$ which, moreover, belongs to $m_0(a)$ ⁽⁷⁾. Now, it is clear that T must coincide with the tangent plane $T_y \partial \mathbb{B}(a, d_0(a))$.

To be a trifle more precise as far as the existence of the limits is concerned, let us recall (cf. [4]) that any definable function $f: (0, 1) \rightarrow \mathbb{R}^n$ with bounded image extends by continuity onto $[0, 1]$ ⁽⁸⁾. We apply this result to the two definable, bounded functions $\mu(t) = m_t(a)$ (its definability follows from [8] and the boundedness comes from the inclusion $\limsup_{t \rightarrow 0+} m_t(a) \subset m_0(a)$ as a consequence of the graphical convergence ⁽⁹⁾ of m_t to m_0 cf. [11]) and $\tau(t) = T_{m_t(a)} \partial \mathbb{B}(a, d_t(a))$. As for the latter, we observe that $\tau(t)$ is a hyperplane and as such can be represented by the normal vector $(a - m_t(a))/d_t(a)$ whose dependence on t is clearly definable, for $m_t(a)$ and the distance $d_t(a)$ are both definable functions of t . Of course, the vectors converge to $(a - \lim m_t(a))/d_0(a)$ and this vector defines T .

Once we have established this, we observe that $a \in M_0$ implies that

$$\sup\{s \geq 1 \mid \mathbb{B}(y + s(a - y)), sd_0(a)) \cap X_0 = \emptyset\} = 1$$

where y is the point defined earlier as the limit of $\mu(t)$. If we succeed in showing that $r_0 = 1$, we are done, as it means that $U \cap C_t \neq \emptyset$, for t close to zero, where C_t denotes the central set of X_t ⁽¹⁰⁾. Indeed, the points $m_t(a) + r(t)(a - m_t(a))$ certainly belong to the central sets C_t and we know that $M_t \subset C_t \subset \overline{M_t}$ (and the closures do not alter the limit). This gives the desired contradiction.

Why is $r_0 = 1$? Observe that by definition $r(t) \geq 1$, whence $r_0 \geq 1$. Suppose that $r_0 > 1$. This means that for some constant $c > 0$ we have for all t close to zero,

$$\mathbb{B}(m_t(a) + (1 + c)(a - m_t(a)), (1 + c)d_t(a)) \cap X_t = \emptyset$$

These balls converge to the closure of the ball

$$B := \mathbb{B}(y + (1 + c)(a - y), (1 + c)d_0(a))$$

But the latter open ball can contain no point of X_0 , otherwise such a point would be reachable by points from the sets X_t , due to the

⁶Hereafter $G_{n-1}(\mathbb{R}^n)$ denotes the $(n - 1)$ th Grassmannian.

⁷It follows from the graphical convergence of m_t to m_0 .

⁸This can be obtained e.g. as a simple consequence of the Curve Selecting Lemma.

⁹Similarly as is done in [11], it can be shown that $X_t \xrightarrow{K} X_0$ iff we have the graphical convergence of the multifunctions m_t to m_0 , i.e. the graphs $\Gamma_{m_t} \xrightarrow{K} \Gamma_{m_0}$.

¹⁰i.e. the set of the centres of maximal balls contained in $\mathbb{R}^n \setminus X_t$.

convergence, which is clearly impossible. On the other hand, we have $B \supsetneq \mathbb{B}(a, d_0(a))$ which contradicts $a \in M_0$.

The proof is accomplished. \square

Remark 4.6. The proof uses typically definable methods. One could be tempted to prove the Theorem in a more general setting, at least for $k = 1$ (a one-parameter family, not necessarily definable). A natural approach could be the following.

As observed in [11], the convergence $X_t \xrightarrow{K} X_0$ is equivalent to the local uniform convergence of the 1-Lipschitz functions $d_t(x) := \text{dist}(x, X_t)$ to $d_0(x)$. If $a \in M_0$, then $a \notin X_0$ and so it is a non-differentiability point of d_0 , cf. [13] and [1] ⁽¹¹⁾. In order to argue by contradiction, suppose that $a \notin \liminf M_t$, i.e. all the functions d_t for $t \neq 0$ close enough to zero are differentiable (and so of class \mathcal{C}^1 , cf. [1]) in a common neighbourhood $U \ni a$.

The sequence of gradients (∇d_t) is uniformly bounded by 1. If we could prove that for some sequence $t_\nu \rightarrow 0$ it is also equicontinuous, then by the Arzelà-Ascoli Theorem, we could extract a subsequence convergent to some function g in U (or in a possibly smaller neighbourhood of a). But as $d_t \rightarrow d_0$, it would follow from classical analysis that $g = \nabla d_0$, i.e. d_0 would be \mathcal{C}^1 near a , contrary to the assumptions.

As we can see, the main problem lies in proving the equicontinuity of (∇d_{t_ν}) , although we know that $\nabla d_t(x) = (1/d_t(x))(x - m_t(x))$ (see e.g. [1]) and the convergence of X_t to X_0 is equivalent to the convergence of the graphs Γ_{m_t} to Γ_{m_0} (see [11]) ⁽¹²⁾. Unfortunately, it quickly turns out that at some point or another we would need to know that m_0 is univalent which is precisely the assertion we want to prove. This kind of problem seems to be unavoidable even if we try to use the much more general approach presented in [11] (a generalized Arzelà-Ascoli property for multifunctions). Actually, by [11] 5.40 it would be sufficient to prove the upper semicontinuity of m_t at a , for it implies the convergence of $m_t(a)$ to $m_0(a)$ which shows that the latter must consist of one point, if the m_t 's are univalent. Again, it seems to be impossible to do it without already knowing that m_0 is univalent.

The question whether some version of our main result is true without the definability assumption remains open.

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¹¹It is shown in [1] Theorem 2.21 that $x \in M_X$ iff x is neither a point of X , nor of the set of differentiability points of the distance function.

¹²By the way, our fifth footnote stems out from these considerations on the convergence of d_t .

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